

Convergence of the Bernstein-Durrmeyer operators in variation seminorm

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Abstract. The aim of this paper is to study variation detracting property and convergence in variation of the Bernstein-Durrmeyer modifications of the classical Bernstein operators in the space of functions of bounded variation. These problems are studied with respect to the variation seminorm. Moreover we also study the problem of the rate of approximation.

Keywords: Approximation in variation, absolutely continuous functions, Bernstein-Durrmeyer operators.

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1. Introduction

The classical Bernstein operators, defined for functions $f \in C[0, 1]$, are of the form

$$(B_n f)(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{n,k}(x), \quad n \in \mathbb{N} \quad (1)$$

where $p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis function, in which the function f can be completely reconstructed for all $x \in [0, 1]$ from its sampled value $f(k/n)$ taken at the nodes k/n ($k = 0, 1, \dots, n$).

Let $L_1 = L_1[0, 1]$ be the space of all Lebesgue measurable and integrable complex valued functions defined on $[0, 1]$, endowed with the usual norm. In order to find a positive answer to the approximation problem for Lebesgue

integrable functions defined on the interval $[0, 1]$, J. L. Durrmeyer [1] and, independently Lupas [2] introduced the integral modifications of the well-known Bernstein polynomials, called Bernstein-Durrmeyer operators. These are defined as

$$(D_n f)(x) = (n+1) \sum_{k=0}^n p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt, \quad x \in [0, 1]. \quad (2)$$

Here, the sampled value $f(k/n)$ in (1) is replaced by

$$(n+1) \int_0^1 p_{n,k}(t) f(t) dt.$$

In 1981 Derriennic [3] first studied these operators in details. After that, Bernstein-Durrmeyer operators have been extensively studied by Gonska and Zhou [4], Ditzian and Ivanov [5], and several other authors.

The present work is strongly motivated by the paper [6], in which the authors have introduced, developed in details and studied the deep interconnections between variation detracting property and the convergence in variation for Bernstein-type polynomials and singular convolution integrals. After this fundamental study, the convergence in variation seminorm has become a new research field in the theory of approximation. One should also note the results obtained in this direction in [[8]-[12]].

The aim of the paper is to establish variation detracting property and convergence in variation of Bernstein-Durrmeyer operators in the space of functions of bounded variation. The rate of approximation is given with respect to the variation seminorm.

2. Notation and Preliminaries

For the notation; let I be a bounded or unbounded interval. Throughout the work, $V_I[f]$ stands for the total Jordan variation of the real-valued function f defined on I . We dealt with the classes $TV(I)$ and $BV(I)$ of all the functions of bounded variation on $I \subset \mathbb{R}$, endowed with the seminorm and norm, respectively

$$\|f\|_{TV(I)} := V_I[f],$$

and

$$\|f\|_{BV(I)} := V_I[f] + |f(c)|,$$

where c is any fixed point of I . Some interesting properties of the space $TV(I)$ are presented in [6].

In order to obtain a convergence result in the variation seminorm, it is necessary and important to state the variation detracting property. It was G. G. Lorentz who showed in his candidate thesis [7] that the operators B_n have the property

$$V_{[0,1]}[B_n f] \leq V_{[0,1]}[f] \quad (n \in \mathbb{N}),$$

called the variation detracting or variation diminishing property, i.e., positive linear operators from the space of functions of bounded variation into itself do not increase the total variation of functions.

Set $AC(I)$, the space of all absolutely continuous real-valued functions on I , is a closed subspace of $TV(I)$ with respect to the convergence induced by the seminorm $\|f\|_{TV(I)}$. In addition, it is well known that is $\lim_{n \rightarrow \infty} V_I[g_n - g] = 0$ for a sequence $(g_n)_{n \geq 1}$ in $AC(I)$, then also $g \in AC[0, 1]$ and

$$V_I[g_n - g] = \int_I |g'_n(t) - g'(t)| dt.$$

So, convergence in variation of $(g_n)_{n \geq 1} \subset AC(I)$ to g , exactly means the convergence of the derivatives g'_n to g' in the $L_1(I)$ -norm.

For the proof of theorems in the following sections, since $(d/dx)p_{k,n}(x) = (k - nx)p_{k,n}(x)/X$, we calculate the two fundamental representations for derivative of $(D_n f)(x)$ as,

$$(D_n f)'(x) = \frac{(n+1)}{X} \sum_{k=0}^n (k - nx) p_{n,k}(x) \int_0^1 f(t) p_{n,k}(t) dt \quad (3)$$

and

$$(D_n f)'(x) = n \sum_{k=0}^{n-1} p_{n-1,k}(x) (n+1) \int_0^1 f(t) [p_{n,k+1}(t) - p_{n,k}(t)] dt \quad (4)$$

where $X = x(1-x)$. In the same proofs we need sum moments for the operators (2). Let us define the sum moments for $r = 0, 1, 2, 3, 4$,

$$T_{r,n}(x) = \sum_{k=0}^n k^r p_{n,k}(x). \quad (5)$$

Then there hold

$$T_{r,n}(x) = \begin{cases} 1 & r = 0 \\ nx & r = 1 \\ n(n-1)x^2 + nx & r = 2 \\ n(n-1)(n-2)x^3 + 3n(n-1)x^2 + nx & r = 3 \\ n(n-1)(n-2)(n-3)x^4 + 6n(n-1)(n-2)x^3 + 7n(n-1)x^2 + nx & r = 4. \end{cases} \quad (6)$$

3. Variation Detracting Property of Bernstein Durrmeyer Operators

In this section, we state the variation detracting property of the Bernstein-Durrmeyer Operators.

Theorem 3.1. If $f \in TV[0, 1]$, then

$$V_{[0,1]}[D_n f] \leq V_{[0,1]}[f] \quad (7)$$

and

$$\|D_n f\|_{BV[0,1]} \leq \|f\|_{BV[0,1]} \quad (8)$$

hold true.

Proof. For convenience we write the Bernstein-Durrmeyer operators as;

$$(D_n f)(x) = \sum_{k=0}^n p_{n,k}(x) F_{k,n}$$

where

$$F_{k,n} := (n+1) \int_0^1 f(t) p_{n,k}(t) dt.$$

As in the (4), we calculate differentiation of (2)

$$\begin{aligned}
(D_n f)'(x) &= \sum_{k=0}^n p'_{n,k}(x) F_{k,n} = \sum_{k=1}^n \binom{n}{k} k x^{k-1} (1-x)^{n-k} F_{k,n} \\
&\quad - \sum_{k=0}^{n-1} \binom{n}{k} x^k (n-k) (1-x)^{n-k-1} F_{k,n} \\
&= n \sum_{k=0}^{n-1} p_{n-1,k}(x) F_{k+1,n} - n \sum_{k=0}^{n-1} p_{n-1,k}(x) F_{k,n} \\
&= n \sum_{k=0}^{n-1} p_{n-1,k}(x) [F_{k+1,n} - F_{k,n}] \\
&= n \sum_{k=0}^{n-1} p_{n-1,k}(x) \Delta F_{k,n}. \tag{9}
\end{aligned}$$

Considering the representation (9) of $(D_n f)'$, one has

$$\begin{aligned}
\|D_n f\|_{TV[0,1]} &= V_{[0,1]}[D_n f] = \int_0^1 |(D_n f)'(x)| dx \\
&\leq n \sum_{k=0}^{n-1} |\Delta F_{k,n}| \int_0^1 p_{n-1,k}(x) dx.
\end{aligned}$$

Since $n \int_0^1 p_{n-1,k}(x) dx = 1$, we get

$$\|D_n f\|_{TV[0,1]} \leq \sum_{k=0}^{n-1} |\Delta F_{k,n}|. \tag{10}$$

Here $\Delta F_{k,n}$ is

$$\begin{aligned}
\Delta F_{k,n} &= (n+1) \int_0^1 f(t) [p_{n,k+1}(t) - p_{n,k}(t)] dt \\
&= (n+1) \int_0^1 f(t) \Delta p_{n,k}(t) dt.
\end{aligned}$$

Since

$$\begin{aligned}
p'_{n,k}(x) &= n(p_{n-1,k-1}(x) - p_{n-1,k}(x)) \\
&= -n(p_{n-1,k}(x) - p_{n-1,k-1}(x)) \\
&= -n\Delta p_{n-1,k-1}(x),
\end{aligned}$$

and

$$\Delta p_{n-1,k-1}(t) = \frac{-p'_{n,k}(t)}{n} \implies \Delta p_{n,k}(t) = \frac{-p'_{n+1,k+1}(t)}{n+1},$$

we get

$$|\Delta F_{k,n}| = \left| (n+1) \int_0^1 f(t) \left[\frac{-p'_{n+1,k+1}(t)}{n+1} \right] dt \right|. \quad (11)$$

So from (10) and (11), we obtain

$$\begin{aligned}
V_{[0,1]}[D_n f] &= \sum_{k=0}^{n-1} |\Delta F_{k,n}| = \sum_{k=0}^{n-1} \left| - \int_0^1 f(t) p'_{n+1,k+1}(t) dt \right| \\
&= \sum_{k=0}^{n-1} \left| \int_0^1 p_{n+1,k+1}(t) f'(t) dt \right| \\
&\leq \sum_{k=0}^{n-1} \int_0^1 p_{n+1,k+1}(t) |f'(t)| dt \\
&= \int_0^1 \sum_{k=0}^{n-1} \binom{n+1}{k+1} t^{k+1} (1-t)^{n-k} |f'(t)| dt \\
&= \int_0^1 \sum_{k=1}^n \binom{n+1}{k} t^k (1-t)^{n+1-k} |f'(t)| dt \\
&\leq \int_0^1 \sum_{k=0}^{n+1} \binom{n+1}{k} t^k (1-t)^{n+1-k} |f'(t)| dt \\
&= \int_0^1 (t+1-t)^{n+1} |f'(t)| dt = \int_0^1 |f'(t)| dt \\
&= V_{[0,1]}[f].
\end{aligned}$$

The desired estimate (7) is now evident.

Since $(D_n f)(0) = (n+1) \int_0^1 (1-t)^n f(t) dt$ and $\|f\|_{BV[I]} := V_I[f] + |f(0)|$, relation (8) is a result of (7). Indeed

$$\begin{aligned} \|D_n f\|_{BV[0,1]} &= V_{[0,1]}[D_n f] + |(D_n f)(0)| \\ &\leq V_{[0,1]}[f] + \left| (n+1) \int_0^1 (1-t)^n f(t) dt \right| \end{aligned}$$

Since $f \in TV[0,1]$ and $\left| (n+1) \int_0^1 (1-t)^n f(t) dt \right| = |f(0)| \leq |f(c)|$ where c is any fixed point of $[0,1]$, we get

$$\|D_n f\|_{BV[0,1]} \leq \|f\|_{BV[0,1]}.$$

Thus, the proof of the theorem is complete.

4. Rate of Approximation in TV -norm

This section is deal with the rates of approximation $D_n g$ to g in the variation seminorm.

Theorem 4.1. Let $g'' \in AC[0,1]$, then

$$V_{[0,1]}[D_n g - g] \leq \frac{2(C+1)}{\sqrt{n}} \{V_{[0,1]}[g] + V_{[0,1]}[g'']\}$$

where $C > 1$ is a constant.

Proof. By Taylor's formula with integral remainder term, one has

$$g(t) = g(x) + (t-x)g'(x) + (t-x)^2 \frac{g''(x)}{2} + \frac{1}{2} \int_0^{t-x} (t-x-u)^2 g'''(x+u) du.$$

Substituting $x+u=v$, it is easily reached that

$$g(t) = g(x) + (t-x)g'(x) + (t-x)^2 \frac{g''(x)}{2} + \frac{1}{2} \int_x^t (t-v)^2 g'''(v) dv. \quad (12)$$

If we apply the operator D'_n to both sides of equality (12), one has from (3)

$$(D_n g)'(x) = A_{0,n}(x)g(x) + A_{1,n}(x)g'(x) + A_{2,n}(x)g''(x) + (R_n g)(x) \quad (13)$$

where

$$A_{j,n}(x) = \frac{n+1}{Xj!} \sum_{k=0}^n (k-nx) p_{n,k}(x) \int_0^1 (t-x)^j p_{n,k}(t) dt \quad (j = 0, 1, 2)$$

and the remainder term is given by

$$\begin{aligned} (R_n g)(x) &= \frac{n+1}{2X} \sum_{k=0}^n (k-nx) p_{n,k}(x) \int_0^1 \left[\int_x^t (t-v)^2 g'''(v) dv \right] p_{n,k}(t) dt \\ &\leq \frac{n+1}{2X} \sum_{k=0}^n (k-nx) p_{n,k}(x) \int_0^1 (t-x)^2 \left[\int_x^t g'''(v) dv \right] p_{n,k}(t) dt. \end{aligned} \quad (14)$$

Calculating (5), (6) and using the property of the binomial coefficients, Beta and Gamma functions, we obtain

$$\begin{aligned} A_{0,n}(x) &= \frac{1}{X} [T_{1,n}(x) - nxT_{0,n}(x)] = 0, \\ A_{1,n}(x) &= \frac{1}{(n+2)X} [T_{2,n}(x) + (1-nx)T_{1,n}(x) - nxT_{0,n}(x)] = \frac{n}{n+2} \end{aligned}$$

and

$$\begin{aligned} A_{2,n}(x) &= \left[\frac{T_{3,n}(x) + 3(1-(n+2)x)T_{2,n}(x) + (2-(5n+6)x + 2n(n+3)x^2)T_{1,n}(x)}{2(n+2)(n+3)X} \right. \\ &\quad \left. + \frac{(2-(5n+6)x + 2n(n+3)x^2)T_{1,n}(x) - 2nx(1-(n+3)x)T_{0,n}(x)}{2(n+2)(n+3)X} \right] \\ &= \frac{2n(1-2x)}{(n+2)(n+3)}. \end{aligned}$$

So by (12), (13) and (14), we have

$$(D_n g)'(x) = \frac{n}{n+2}g'(x) + \frac{2n(1-2x)}{(n+2)(n+3)}g''(x) + (R_n g)(x). \quad (15)$$

In order to estimate the integration domain of the double integral in the remainder term (14), we divide the summation into four different sums as following ($[x]$ denotes the integer part of x)

$$(R_n g)(x) = \sum_{i=1}^4 (B_{i,n} g)(x) \equiv \sum_{i=1}^4 B_{i,n} g \quad (16)$$

where

$$\begin{aligned} B_{1,n} g &= \frac{1}{2X} \sum_{k=0}^{[nx]} (k - nx) p_{n,k}(x) (n+1) \int_0^{k/n} (t-x)^2 \left[\int_x^t g'''(v) dv \right] p_{n,k}(t) dt, \\ B_{2,n} g &= \frac{1}{2X} \sum_{k=0}^{[nx]} (k - nx) p_{n,k}(x) (n+1) \int_{k/n}^x (t-x)^2 \left[\int_x^t g'''(v) dv \right] p_{n,k}(t) dt, \\ B_{3,n} g &= \frac{1}{2X} \sum_{k=[nx]+1}^n (k - nx) p_{n,k}(x) (n+1) \int_x^{k/n} (t-x)^2 \left[\int_x^t g'''(v) dv \right] p_{n,k}(t) dt, \end{aligned}$$

and

$$B_{4,n} g = \frac{1}{2X} \sum_{k=[nx]+1}^n (k - nx) p_{n,k}(x) (n+1) \int_{k/n}^1 (t-x)^2 \left[\int_x^t g'''(v) dv \right] p_{n,k}(t) dt.$$

Now we estimate $B_{i,n} g$ for $i = 1, 2, 3, 4$, respectively. Let us estimate $B_{1,n} g$

as follows

$$\begin{aligned}
|B_{1,ng}| &\leq \frac{1}{2X} \sum_{k=0}^{[nx]} |k - nx| p_{n,k}(x) (n+1) \int_0^{k/n} (t-x)^2 \left| \int_x^t g'''(v) dv \right| p_{n,k}(t) dt \\
&= \frac{1}{2X} \sum_{k=0}^{[nx]} (k - nx) p_{n,k}(x) (n+1) \int_0^{k/n} (t-x)^2 \left[\int_x^t |g'''(v)| dv \right] p_{n,k}(t) dt \\
&\leq \frac{1}{2X} \sum_{k=0}^{[nx]} (k - nx) p_{n,k}(x) (n+1) x^2 \int_0^{k/n} \left[\int_x^t |g'''(v)| dv \right] p_{n,k}(t) dt \\
&= \frac{1}{2X} \sum_{k=0}^{[nx]} (nx - k) p_{n,k}(x) (n+1) x^2 \int_0^{k/n} \left[\int_t^x |g'''(v)| dv \right] p_{n,k}(t) dt \\
&\leq \frac{1}{2X} \sum_{k=0}^{[nx]} (nx - k) p_{n,k}(x) (n+1) x^2 \int_0^{k/n} \left[\int_0^1 |g'''(v)| dv \right] p_{n,k}(t) dt \\
&\leq \frac{1}{2X} \|g'''\| \sum_{k=0}^{[nx]} (nx - k) p_{n,k}(x) (n+1) x^2 \int_0^1 p_{n,k}(t) dt \\
&= \frac{x}{2(1-x)} \|g'''\| \sum_{k=0}^{[nx]} (nx - k) p_{n,k}(x) \leq \frac{x}{2(1-x)} \|g'''\| [nxT_{0,n}(x) - T_{1,n}(x)]
\end{aligned}$$

where

$$\|f\| := \|f\|_{L_1(0,1)}.$$

In view of (6) we obtain

$$\|B_{1,ng}\| = 0. \quad (17)$$

Analogously, $B_{2,ng}$ can be estimated by

$$|B_{2,ng}| \leq \frac{1}{2X} \sum_{k=0}^{[nx]} |k - nx| p_{n,k}(x) (n+1) \int_{k/n}^x (t-x)^2 \left| \int_x^t |g'''(v)| dv \right| p_{n,k}(t) dt$$

$$\begin{aligned}
&= \frac{1}{2X} \sum_{k=0}^{[nx]} (k - nx) p_{n,k}(x) (n+1) \int_{k/n}^x (t-x)^2 \left[\int_x^t |g'''(v)| dv \right] p_{n,k}(t) dt \\
&\leq \frac{1}{2X} \sum_{k=0}^{[nx]} (k - nx) p_{n,k}(x) (n+1) \left(\frac{k}{n} - x \right)^2 \int_{k/n}^x \left[\int_x^t |g'''(v)| dv \right] p_{n,k}(t) dt \\
&= \frac{1}{2X} \sum_{k=0}^{[nx]} (nx - k) p_{n,k}(x) (n+1) \left(\frac{k}{n} - x \right)^2 \int_{k/n}^x \left[\int_t^x |g'''(v)| dv \right] p_{n,k}(t) dt \\
&\leq \frac{1}{2X} \sum_{k=0}^{[nx]} (nx - k) p_{n,k}(x) (n+1) \left(\frac{k}{n} - x \right)^2 \int_{k/n}^x \left[\int_0^1 |g'''(v)| dv \right] p_{n,k}(t) dt \\
&\leq \frac{\|g'''\|}{2X} \sum_{k=0}^{[nx]} (nx - k) p_{n,k}(x) (n+1) \left(\frac{k}{n} - x \right)^2 \int_{k/n}^x p_{n,k}(t) dt \\
&\leq \frac{\|g'''\|}{2X} \sum_{k=0}^{[nx]} (nx - k) p_{n,k}(x) \frac{(k - nx)^2}{n^2} = \|g'''\| \sum_{k=0}^{[nx]} |k - nx| p_{n,k}(x) \frac{(k - nx)^2}{n^2} \\
&\leq \frac{\|g'''\|}{2n^2 X} \sum_{k=0}^n |k - nx| p_{n,k}^{1/2}(x) (k - nx)^2 p_{n,k}^{1/2}(x).
\end{aligned}$$

by using Hölder inequality and (6)

$$\begin{aligned}
&\leq \frac{\|g'''\|}{2n^2 X} \left(\sum_{k=0}^n (k - nx)^2 p_{n,k}(x) \right)^{1/2} \left(\sum_{k=0}^n (k - nx)^4 p_{n,k}(x) \right)^{1/2} \\
&= \frac{\|g'''\|}{2n^2 X} \left(T_{2,n}(x) - 2nxT_{1,n}(x) + (nx)^2 T_{0,n}(x) \right)^{1/2} \\
&\quad \times \left(T_{4,n}(x) - 4nxT_{3,n}(x) + 6(nx)^2 T_{2,n}(x) - 4(nx)^3 T_{1,n}(x) + (nx)^4 T_{0,n}(x) \right)^{1/2} \\
&= \frac{\|g'''\|}{2n^2 X} (nX)^{1/2} \left(3(nX)^2 + (1 - 6X)nX \right)^{1/2} \\
&= \frac{\|g'''\|}{2n} (3nX + 1 - 6X)^{1/2}
\end{aligned}$$

So if we take norm of $B_{2,n}g$, we get

$$\|B_{2,n}g\| \leq \frac{1}{\sqrt{n}} \|g'''\|. \quad (18)$$

Now we estimate $B_{3,n}g$ with (6) as following

$$\begin{aligned}
|B_{3,n}g| &\leq \frac{1}{2X} \sum_{k=[nx]+1}^n |k-nx| p_{n,k}(x) (n+1) \int_x^{k/n} (t-x)^2 \left| \int_x^t |g'''(v)| dv \right| p_{n,k}(t) dt \\
&= \frac{1}{2X} \sum_{k=[nx]+1}^n (k-nx) p_{n,k}(x) (n+1) \left(\frac{k}{n} - x \right)^2 \int_x^{k/n} \left[\int_x^t |g'''(v)| dv \right] p_{n,k}(t) dt \\
&\leq \frac{\|g'''\|}{2n^2 X} \sum_{k=0}^n (k-nx)^3 p_{n,k}(x) \\
&= \frac{\|g'''\|}{2n^2 X} [T_{3,n}(x) - 3nxT_{2,n}(x) + 3(nx)^2 T_{1,n}(x) - (nx)^3 T_{0,n}(x)] \\
&= \frac{\|g'''\|}{2n} (1-2X).
\end{aligned}$$

So, we get the following inequality for the norm of $B_{3,n}g$

$$\|B_{3,n}g\| \leq \frac{1}{2n} \|g'''\|. \quad (19)$$

Finally, we estimate $B_{4,n}g$ with the (6)

$$\begin{aligned}
|B_{4,n}g| &\leq \frac{1}{2X} \sum_{k=[nx]+1}^n |k-nx| p_{n,k}(x) (n+1) \int_{k/n}^1 (t-x)^2 \left| \int_x^t |g'''(v)| dv \right| p_{n,k}(t) dt \\
&= \frac{1}{2X} \sum_{k=[nx]+1}^n (k-nx) p_{n,k}(x) (n+1) (1-x)^2 \int_{k/n}^1 \left[\int_0^1 |g'''(v)| dv \right] p_{n,k}(t) dt \\
&\leq \frac{1}{2X} \|g'''\| \sum_{k=[nx]+1}^n (k-nx) p_{n,k}(x) (1-x)^2 \\
&\leq \frac{1-x}{2x} \|g'''\| [T_{1,n}(x) - nxT_{0,n}(x)] = 0,
\end{aligned}$$

and hence

$$\|B_{4,n}g\| = 0. \quad (20)$$

Collecting the results in (17)-(20), we have by (16)

$$\|R_n g\| \leq \frac{1}{\sqrt{n}} \|g'''\| + \frac{1}{2n} \|g'''\| \leq \frac{2}{\sqrt{n}} \|g'''\|.$$

Finally we obtain by using (15)

$$\|(D_n g)' - g'\| \leq \frac{2}{n+2} \|g'\| + \frac{2}{n+2} \|g''\| + \frac{2}{\sqrt{n}} \|g'''\|.$$

According to Stein's inequality (see, e.g., [13], Theorem A10.1) one has

$$\begin{aligned} \|g''(x)\|_{L_1(0,1)} &\leq C \sqrt{\|g'(x)\|_{L_1(0,1)} \|g'''\|_{L_1(0,1)}} \\ &\leq C \left(\|g'\|_{L_1(0,1)} + \|g'''\|_{L_1(0,1)} \right), \end{aligned}$$

where $C > 1$. For $\frac{1}{n+2} < \frac{1}{\sqrt{n}}$ we have

$$\|(D_n g)' - g'\| \leq \frac{2(C+1)}{\sqrt{n}} (\|g'\| + \|g'''\|).$$

So the proof is now complete.

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